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Mathematical calculation for exact solutions of infinitely strongly interacting Fermi gases in tight waveguides

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Abstract

Exact analytical solutions of the fundamental systems of N quasi-one-dimensional spin-1/2 fermions with infinite delta repulsion in an arbitrary confining potential were presented in our previous letter (Guan *et al* 2009 *Phys. Rev. Lett.* **102** 160402). The solutions are the simultaneous eigenstates of the Hamiltonian H and the total spin operators S^2 and S_z, which fulfil Girardeau's hard-core contacting boundary condition and are antisymmetric under odd permutations among fermions. They are approximate solutions when the coefficient in the delta repulsion is large but finite, and according to the Lieb–Mattis theorem, the solution with the lowest S is the ground state for the system. A detailed mathematical calculation for the exact solutions is presented. The property of the spin-dependent reduced one-body density is discussed.

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1. Introduction

Recently, one-dimensional (1D) strongly correlated atomic systems have attracted extensive theoretical and experimental attention due to experimental progress in manipulating cold atoms in effective 1D waveguides [2, 3]. For effective 1D systems, confinement-induced resonance [4, 5] allows Feshbach resonance tuning of the effective 1D interactions to the very strongly interacting regime where correlation effects are greatly enhanced [6–8]. The Tonks–Girardeau (TG) gas, which is the Bose gas in the strongly interacting limit, has been experimentally realized [9, 10]. The Fermi gas in the unitary limit (the infinitely interaction limit) can be also produced using magnetic field-induced Feshbach resonances [11–13]. More recently, an interacting 1D Fermi gas in a two-dimensional optical lattice with tunable interaction strengths by Feshbach resonance has also been experimentally realized by Moritz *et al* [14], which offers the opportunity of studying the 1D interacting Fermi gases even in the strong interaction limit.

Exact solutions and methods capable of dealing with strong correlations have played an especially important role in understanding the physical properties of the 1D quantum gas in the strongly interacting limit [15–19]. Despite the elegant method of Bose–Fermi mapping having existed since 1960, generalization to systems including the spin degree of freedom is a highly non-trivial problem and was only recently tackled for mixtures of Bose and Fermi gases and the spinor Bose gas [20, 21]. The *indistinguishable* spin-1/2 Fermi gas in the TG limit has been studied in our recent work [1] and the exact analytical constructions of ground state wavefunctions for fundamental systems of N quasi-one-dimensional spin-1/2 fermions with infinite delta repulsion in an arbitrary confining potential are given there. The wavefunctions presented in our previous letter [1] are the simultaneous eigenstates of the Hamiltonian H and the total spin operators S^2 and S_z, which fulfil Girardeau's hard-core contacting boundary condition and are antisymmetric under odd permutations among fermions. It is our purpose in this paper to give some examples of the solutions and the general mathematical calculation in some detail.

Consider a general system of N *indistinguishable* spin-1/2 fermions in an elongated potential trap with $\omega_\perp \gg \omega_x$, where ω_x and $\omega_\perp \equiv \omega_y = \omega_z$ are angular frequencies in the axial and radial directions, respectively. Under the condition $\omega_\perp/\omega_x \gg N$, Fermi systems are dynamically described by an effective 1D Hamiltonian:

$$H = \sum_{i=1}^{N} H_i + g_{1d} \sum_{i<j} \delta(x_i - x_j), \qquad H_i = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + V(x_i), \tag{1}$$

where $g_{1d} = -2\hbar^2/(ma_{1d})$ is the effective 1D interaction strength related to the three-dimensional s-wave scattering length a_s by $a_{1d} = -l_\perp(l_\perp/a_s - |\zeta(1/2)|/\sqrt{2})$ with $l_\perp = \sqrt{\hbar/m\omega_\perp}$, the characteristic oscillator length in the radial direction [4, 5]. $V(x_i)$ is an arbitrary confining potential, say, $V(x_i) = m\omega_x^2 x_i^2/2$ for a harmonic potential. Interacting spin-1/2 fermion systems have been intensively studied [22–27]; however, there are few rigorous results except for the homogenous Yang–Gaudin model [15, 16].

The general contacting boundary condition for a TG Fermi gas in the strongly interacting limit can be represented as

$$\Psi(x_i = x_j) = 0. \tag{2}$$

Then, the wave function fulfilling the above boundary condition is composed of the Slater determinant of N orthonormal orbitals $\phi_1(x), \ldots, \phi_N(x)$,

$$\psi_A(x_1, \ldots, x_N) = (N!)^{-1/2} \det[\phi_j(x_i)]_{i=1,\ldots,N}^{j=1,\ldots,N}, \tag{3}$$

where $\phi_j(x_i)$ is the eigenfunction of the single particle Hamiltonian. Since H is spin independent, it is commutable with the total spin operator $\hat{\boldsymbol{S}} = \sum_i \hat{\boldsymbol{S}}_i$, where $\hat{\boldsymbol{S}}_i$ is the spin operator of the ith particle. This implies that the system possesses a global $SU(2)$ symmetry such that the eigenstates of H are simultaneously eigenstates of $\hat{\boldsymbol{S}}^2$ and $\hat{S}_z$ and only the eigenstates with the largest eigenvalue $S_z = S$ need to be considered. The remaining eigenstates can be calculated from them by the lowering operator $\hat{S}_-$. In addition, the total wave function of N *indistinguishable* fermions has to be antisymmetric under transposition of any two particles.

According to (3), the ground state corresponds to the fully filled state with the lowest N orbitals and excited states are generated by occupying higher orbitals. Similar to the spinor boson case, the ground state is highly degenerate in the TG limit due to the different spin configurations. Among the family of degenerate ground states, the ferromagnetic spin state with $S_z = S = N/2$ is a product of all spins up which is totally symmetric in its permutations. The total wave function, antisymmetric under transpositions, takes a factorized form

$$\Psi = \psi_A(x_1, \ldots, x_N)\chi_1(1)\cdots\chi_1(N), \tag{4}$$

where $\chi_1(i)$ denotes the up-spin state of the ith particle.

According to the Lieb–Mattis theorem [17], for a finite interaction strength, the energy of the ground state with a given S is lower than that with a higher S. A simple example [18] shows that the Lieb–Mattis theorem holds even for a system with a delta repulsion whose coefficient is large but finite. Obviously, the product of the spatial wave function (3) and the spinor function with $S < N/2$ does not fulfil the requirement of antisymmetry for an *indistinguishable* Fermi system. In our previous letter [1], we present a wave function ψ formally written as a product of ψ_A and ψ_S, where ψ_A is given in (3) and ψ_S is composed of a linear combination of product of sign functions and spinor functions and is totally symmetric under permutations among particles,

$$\psi_S = \left\{\sum_{\alpha=1}^{N!/(n!m!)} P_\alpha\right\}\left\{Q_1\left(\mathcal{Y}_1^{[n,m]}Z_1\right)\right\} = \sum_{\alpha=1}^{N!/(n!m!)}\left\{\mathcal{Y}_\alpha^{[n,m]}Q_\alpha\right\}Z_\alpha. \tag{5}$$

The notation will be explained in the text. For a system with an infinite delta repulsion in an arbitrary confining potential, these solutions are the simultaneous eigenstates of the Hamiltonian H and the total spin operators S^2 and S_z, which fulfil Girardeau's hard-core contacting boundary condition and are antisymmetric under odd permutations among fermions. They are approximate solutions when the coefficient in the delta repulsion is large but finite, and according to the Lieb–Mattis theorem, the solution with the lowest S is the ground state for the system.

In this paper, we will construct the symmetric wave function ψ_S by group theory in some detail. The spin part of ψ_S is calculated by the method of Young operators in section 2. For a given spin $S_z = S = N/2 - m$, the spinor functions belong to the irreducible representation $[N-m, m]$ of the permutation group S_N so that the spatial functions of ψ_S have to belong to the same representation of S_N. The spatial wave functions are constructed by the sign functions in section 3. ψ_S is composed of the products of spinor functions and spatial functions through the Clebsch–Gordan coefficients of S_N. Three examples for the combinations by the Clebsch–Gordan coefficients are given in sections 4–6. Then, the general form (5) of ψ_S is proved in section 7. In section 8, the total density function of the ground state is calculated to be equal to that of the N-fermion ferromagnetic system. Other possible solutions of ψ_S are discussed in section 9. A summary is given in section 10.

2. Spinor functions

The tensor space $\mathcal{T}$ of rank N with respect to $SU(2)$ can be decomposed into irreducible subspaces by the standard Young operators $\mathcal{Y}_\mu^{[N-m,m]}$ (see section 8.1 in [28]):

$$\mathcal{T} = \bigoplus_{m=0}^{\ell}\bigoplus_{\mu=1}^{d_m}\mathcal{T}_\mu^{[N-m,m]}, \qquad \mathcal{T}_\mu^{[N-m,m]} = \mathcal{Y}_\mu^{[N-m,m]}\mathcal{T}, \tag{6}$$

where $\ell = N/2$ when N is even and $\ell = (N-1)/2$ when N is odd. Hereafter, we define $n = N - m \geqslant m$ for convenience. d_m is the dimension of the representation $[n, m]$ of the permutation group S_N (see (6.22) in [28]):

$$d_m = d_{[n,m]}(S_N) = \frac{N!(n-m+1)}{m!(n+1)!}. \tag{7}$$

The irreducible tensor subspace $\mathcal{T}_\mu^{[n,m]}$ corresponds to the representation $[n, m]$ of $SU(2)$ so that the total spin is $S = N/2 - m = (n-m)/2$. The smallest standard Young operator $\mathcal{Y}_1^{[n,m]}$ with the Young pattern $[n, m]$ corresponds to the Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & \ldots & m & m+1 & \ldots & n \\ \hline n+1 & \ldots & N \\ \cline{1-3}\end{array}\,, \tag{8}$$

and is defined as (see (6.23) in [28])

$$\mathcal{Y}_1^{[n,m]} = \left(\sum_{R\in S_n} R\right)\left(\sum_{T\in S_m} T\right)\left\{\prod_{j=1}^{m}[E - (j\ n+j)]\right\}, \tag{9}$$

where E is the identical permutation and $(j\ n+j)$ is the transposition between j and $(n+j)$. $H_{nm} \equiv S_n \otimes S_m$ is a subgroup of S_N, where S_n and S_m are the permutation groups of the first n objects and the last m objects, respectively. Recall that H_{nm} is not an invariant subgroup of S_N.

The basis tensor $\boldsymbol{\theta}_{\sigma_1\ldots\sigma_N}$, $\sigma_i = 1$ or 2, in $\mathcal{T}$ is just the spinor function and is expressed as the direct product of N two-component spinors $\chi_{\sigma_i}(i)$:

$$\boldsymbol{\theta}_{\sigma_1\ldots\sigma_N} = \chi_{\sigma_1}(1)\ldots\chi_{\sigma_N}(N), \qquad \chi_1(i) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_2(i) = \begin{pmatrix}0\\1\end{pmatrix}. \tag{10}$$

The basis tensor $\mathcal{Y}_\mu^{[n,m]}\boldsymbol{\theta}_{\sigma_1\ldots\sigma_N}$ in $\mathcal{T}_\mu^{[n,m]}$ is the linear combination of the spinor functions (10) which is usually denoted by a tensor Young tableau (see p 358 in [28]). The tensor Young tableau of $\mathcal{Y}_\mu^{[n,m]}\boldsymbol{\theta}_{\sigma_1\ldots\sigma_N}$ in $\mathcal{T}_\mu^{[n,m]}$ is a tableau with the Young pattern $[n, m]$ where the box filled with j in the Young tableau $\mathcal{Y}_\mu^{[n,m]}$ is now filled with σ_j. The spinor function $\mathcal{Y}_1^{[n,m]}Z_1$ where

$$Z_1 = \chi_1(1)\ldots\chi_1(n)\chi_2(n+1)\ldots\chi_2(N) \tag{11}$$

is the spinor function in $\mathcal{T}_1^{[n,m]}$ with $S = S_z = N/2 - m$ which is denoted by the tensor Young tableau with the Young pattern $[n, m]$ where each box in the first line is filled with number 1 and that in the second line is filled with number 2:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & \ldots & 1 & 1 & \ldots & 1 \\ \hline 2 & \ldots & 2 \\ \cline{1-3}\end{array}\,. \tag{12}$$

The tensor Young tableau (12) in $\mathcal{T}_\mu^{[n,m]}$ denotes the spinor function $R_{\mu 1}\mathcal{Y}_1^{[n,m]}Z_1$ with $S = S_z = N/2 - m$, where $R_{\mu 1}$ is the permutation transforming the standard Young tableau $\mathcal{Y}_1^{[n,m]}$ to the standard Young tableau $\mathcal{Y}_\mu^{[n,m]}$ such that $\mathcal{Y}_\mu^{[n,m]} = R_{\mu 1}\mathcal{Y}_1^{[n,m]}R_{\mu 1}^{-1}$ (see section 6.3 in [28]). The d_m spinor functions $R_{\mu 1}\mathcal{Y}_1^{[n,m]}Z_1$ span the representation space of the representation $[n, m]$ of S_N.

$\mathcal{Y}_1^{[n,m]}Z_1^{(\tau)}$ is the spinor function in $\mathcal{T}_1^{[n,m]}$ with $S = N/2 - m$ and $S_z = S - \tau$, $0 \leqslant \tau \leqslant 2S$, where the spinor function $Z_1^{(\tau)}$ contains $N_1 = n - \tau$ two-component upspinors $\chi_1(i)$ and $N_2 = m + \tau$ two-component downspinors $\chi_2(i)$:

$$Z_1^{(\tau)} = \chi_1(1)\ldots\chi_1(n-\tau)\chi_2(n-\tau+1)\ldots\chi_2(N). \tag{13}$$

$\mathcal{Y}_1^{[n,m]} Z_1^{(\tau)}$ is denoted by the tensor Young tableau with the Young pattern $[n, m]$ where there are τ boxes in the first line filled with number 2:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & \ldots & 1 & 1 & \ldots & 1 & 2 & \ldots & 2 \\ \hline 2 & \ldots & 2 \\ \cline{1-3}\end{array} \ . \tag{14}$$

The tensor Young tableau (14) in $\mathcal{T}_\mu^{[n,m]}$ denotes the spinor function $R_{\mu 1}\mathcal{Y}_1^{[n,m]} Z_1^{(\tau)}$ with $S = N/2 - m$ and $S_z = S - \tau$. The d_m spinor functions $R_{\mu 1}\mathcal{Y}_1^{[n,m]} Z_1^{(\tau)}$ also span the representation space of the representation $[n, m]$ of S_N.

Interchanging $1 \leftrightarrow 2$ for all subscripts σ_i of $\chi_{\sigma_i}(i)$ in (13), one obtains $\overline{Z}_1^{(\tau)}$,

$$\overline{Z}_1^{(\tau)} = \chi_2(1) \ldots \chi_2(n-\tau)\chi_1(n-\tau+1) \ldots \chi_1(N). \tag{15}$$

Recall that $N - (n - \tau) = m + \tau = n - (2S - \tau)$. Letting

$$Z_1^{(2S-\tau)} = \chi_1(1) \ldots \chi_1(m+\tau)\chi_2(m+\tau+1) \ldots \chi_2(N), \tag{16}$$

we obtain from the property of Young operators (see section 6.2.4 and section 8.1.2 of [28])

$$\mathcal{Y}_1^{[n,m]} \overline{Z}_1^{(\tau)} = (-1)^m \mathcal{Y}_1^{[n,m]} Z_1^{(2S-\tau)}. \tag{17}$$

$\mathcal{Y}_1^{[n,m]} Z_1^{(2S-\tau)}$ is the spinor function in $\mathcal{T}_1^{[n,m]}$ with $S = N/2 - m$ and $S_z = -S + \tau$.

3. Spatial functions

It is convenient to choose the spatial part of the symmetric wave function ψ_S such that it is composed of the sign functions

$$\mathrm{sgn}_{ij} \equiv \mathrm{sgn}(x_i - x_j) = \frac{x_i - x_j}{|x_i - x_j|}, \tag{18}$$

which satisfy the Laplace equation

$$\left[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right] \mathrm{sgn}(x_i - x_j) = 0. \tag{19}$$

The mixed terms which come from the affection of the Laplace operator on the wave function $\psi = \psi_A \psi_S$ are proportional to the delta functions

$$\begin{aligned}\left[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right] \psi_A \,\mathrm{sgn}(x_i - x_j) &= \mathrm{sgn}(x_i - x_j) \left[\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right] \psi_A \\ &\quad + 2\delta(x_i - x_j) \left[\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right] \psi_A.\end{aligned} \tag{20}$$

Thus, ψ is the eigenstate of the Hamiltonian H when the coefficient g_{1d} of the delta repulsions is infinite.

In order to construct the symmetric wave function ψ_S, its spatial part composed of the sign functions has to belong to the same representation of S_N as the spinor part belongs to. The representation of S_N to which the spinor wave function with $S_z = S = N/2 - m$ belongs is denoted by the Young pattern $[n, m]$.

Define Q_1 as

$$Q_1 \equiv Q_{(n+1),\ldots,N}^{1,2,\ldots,n} = \prod_{i=1}^{n} \prod_{j=n+1}^{N} \mathrm{sgn}(x_i - x_j). \tag{21}$$

Obviously, Q_1 is symmetric with respect to the subgroup $\mathrm{S}_n \otimes \mathrm{S}_m$ of S_N. $\mathcal{Y}_1^{[n,m]}Q_1$ can be calculated straightforwardly and is evidently non-vanishing. Since a Young operator is proportional to a primitive idempotent of S_N, the basis functions $R_{\mu 1}\mathcal{Y}_1^{[n,m]}Q_1$ span the representation space of $[n, m]$ of S_N, where $R_{\mu 1}$ is defined in the previous section (see section 6.3 in [28]). Thus, the totally symmetric wave function ψ_S is

$$\psi_S^{[n,m]} = \sum_{\mu\nu} \left(R_{\mu 1}\mathcal{Y}_1^{[n,m]}Q_1\right)\left(R_{\nu 1}\mathcal{Y}_1^{[n,m]}Z_1\right)\langle [n,m], \mu, [n,m], \nu | [N,0], 1\rangle, \tag{22}$$

where a superscript $[n, m]$ in ψ_S is added for definiteness and $\langle [n,m], \mu, [n,m], \nu | [N,0], 1\rangle$ are the Clebsch–Gordan coefficients of S_N and $[N, 0]$ is the identical representation, which is one-dimensional. Since there is no simple method to calculate the Clebsch–Gordan coefficients of S_N, we will first calculate the symmetric function $\psi_S^{[n,m]}$ from (22) by three examples, and then summarize the general formula for $\psi_S^{[n,m]}$.

4. Representation [4, 1]

We discuss the symmetric function $\psi_S^{[4,1]}$ with $N = 5$ and $S = S_z = 3/2$ as the first example. There are four standard Young tableaux for the Young pattern [4, 1] :

$$\begin{array}{cccc} \mathcal{Y}_1^{[4,1]} & \mathcal{Y}_2^{[4,1]} & \mathcal{Y}_3^{[4,1]} & \mathcal{Y}_4^{[4,1]} \\ \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\ \hline 4 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2 \\ \cline{1-1}\end{array} \end{array} \tag{23}$$

The Young operator $\mathcal{Y}_1^{[4,1]}$ is (see (6.23) in [28])

$$\begin{aligned} \mathcal{Y}_1^{[4,1]} = {} & [E + (1\,2) + (1\,3) + (1\,4) + (2\,3) + (2\,4) + (3\,4) + (1\,2)(3\,4) \\ & + (1\,3)(2\,4) + (1\,4)(2\,3) + (1\,2\,3) + (3\,2\,1) + (1\,2\,4) + (4\,2\,1) \\ & + (1\,3\,4) + (4\,3\,1) + (2\,3\,4) + (4\,3\,2) + (1\,2\,3\,4) + (1\,2\,4\,3) \\ & + (1\,3\,2\,4) + (1\,3\,4\,2) + (1\,4\,2\,3) + (1\,4\,3\,2)][E - (1\,5)]. \end{aligned} \tag{24}$$

The spinor function ϕ_1 with $S_z = S = 3/2$ in the tensor subspace $\mathcal{T}_1^{[4,1]}$ is denoted by the tensor Young tableau (12)

$$\phi_1 = \mathcal{Y}_1^{[4,1]}Z_1 = 24\boldsymbol{\theta}_{11112} - 6\left\{\boldsymbol{\theta}_{11121} + \boldsymbol{\theta}_{11211} + \boldsymbol{\theta}_{12111} + \boldsymbol{\theta}_{21111}\right\}, \tag{25}$$

where $Z_1 = \boldsymbol{\theta}_{11112}$. $\phi_\mu = R_{\mu 1}\phi_1$ is the basis tensor in the tensor subspace $\mathcal{T}_\mu^{[4,1]}$, denoted by the same tensor Young tableau (12). The spinor functions ϕ_μ span the representation space of [4, 1] of S_5. From (23), we have $R_{11} = E$, $R_{21} = (5\,4)$, $R_{31} = (3\,4\,5)$ and $R_{41} = (2\,3\,4\,5)$.

The spatial wave function $\psi_1 = \mathcal{Y}_1^{[4,1]}Q_1$, where $Q_1 = Q_5^{1,2,3,4}$, is

$$\begin{aligned} \psi_1 = \mathcal{Y}_1^{[4,1]}Q_1 &= 24Q_5^{1,2,3,4} - 6\left(Q_4^{1,2,3,5} + Q_3^{1,2,4,5} + Q_2^{1,3,4,5} + Q_1^{2,3,4,5}\right) \\ &= 6[4E - (4\,5) - (3\,5) - (2\,5) - (1\,5)]Q_5^{1,2,3,4}. \end{aligned} \tag{26}$$

$\psi_\mu = R_{\mu 1}\psi_1$ span the representation space of [4, 1] of S_5.

In the orthogonal bases which correspond to the real orthogonal representation $\overline{D}^{[\lambda]}(R)$ of S_N, the Clebsch–Gordan coefficients of S_N for the reduction of $[\lambda] \times [\lambda]$ to the identical representation $[N]$ are always proportional to the Kronecker delta function, because in any permutation R,

$$\sum_{\rho} |\rho\rangle|\rho\rangle \xrightarrow{R} \sum_{\rho\tau\omega} |\tau\rangle|\omega\rangle \overline{D}^{[\lambda]}_{\tau\rho}(R)\overline{D}^{[\lambda]}_{\omega\rho}(R) = \sum_{\tau} |\tau\rangle|\tau\rangle$$

since the representation matrices in the orthogonal bases and in the standard bases, which are calculated by the method of Young operators, are related by a similarity transformation $X_{[\lambda]}$ (see section 6.4 in [28]). The Clebsch–Gordan coefficients in the standard bases can be calculated from those in the orthogonal bases through $X_{[\lambda]}$:

$$\langle[\lambda], \mu, [\lambda], \nu|[N], 1\rangle \propto \sum_{\rho} (X_{[\lambda]})_{\mu\rho}(X_{[\lambda]})_{\nu\rho}. \tag{27}$$

In terms of the method given in problem 22 of chapter 6 in [29], we have

$$X_{[4,1]} = \frac{1}{\sqrt{15}} \begin{pmatrix} \sqrt{15} & 1 & \sqrt{2} & \sqrt{6} \\ 0 & 4 & \sqrt{2} & \sqrt{6} \\ 0 & 0 & 3\sqrt{2} & \sqrt{6} \\ 0 & 0 & 0 & 2\sqrt{6} \end{pmatrix}. \tag{28}$$

Then, neglecting the normalization factor, we have

$$\langle[4,1], \mu, [4,1], \mu|[N], 1\rangle = 2, \qquad \langle[4,1], \mu, [4,1], \nu|[N], 1\rangle = 1, \quad \text{when} \quad \mu \neq \nu. \tag{29}$$

This result holds for all representations $[N-1, 1]$. From (29), we obtain the symmetric wave function $\psi_S^{[4,1]}$ for $N = 5$ and $S_z = S = 3/2$:

$$\psi_S^{[4,1]} = \left(\sum_{\mu=1}^{4} \psi_\mu\right)\left(\sum_{\nu=1}^{4} \phi_\nu\right) + \sum_{\mu=1}^{4} \psi_\mu\phi_\mu.$$

Due to the fundamental property of the Young operators (see (6.30) in [28])

$$(1\,2)\mathcal{Y}_1^{[4,1]} = (1\,2\,3\,4)\mathcal{Y}_1^{[4,1]} = \mathcal{Y}_1^{[4,1]}, \qquad \sum_{\mu=1}^{5} [(5\ \mu)]\,\mathcal{Y}_1^{[4,1]} = 0,$$

we obtain the expression of $\psi_S^{[4,1]}$ by collecting terms with the same $Q_{b_1}^{a_1,a_2,a_3,a_4}$,

$$\begin{aligned} \psi_S^{[4,1]} &= \sum_{\mu=1}^{5} [(5\ \mu)\psi_1]\,[(5\ \mu)\phi_1] \\ &= 6\sum_{\mu=1}^{5} \left[(5\ \mu)Q_5^{1,2,3,4}\right]\left[4(5\ \mu) - \sum_{\nu\neq\mu}(5\ \nu)\right]\phi_1 \\ &= 30\sum_{\mu=1}^{5} (5\ \mu)\left[Q_1\left(\mathcal{Y}_1^{[4,1]}Z_1\right)\right]. \end{aligned} \tag{30}$$

5. Representation [3, 2]

We discuss the symmetric function $\psi_S^{[3,2]}$ with $N = 5$ and $S = S_z = 1/2$ as the second example. There are five standard Young tableaux for the Young pattern [3, 2]:

$$\begin{array}{ccccc} \mathcal{Y}_1^{[3,2]} & \mathcal{Y}_2^{[3,2]} & \mathcal{Y}_3^{[3,2]} & \mathcal{Y}_4^{[3,2]} & \mathcal{Y}_5^{[3,2]} \\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \end{array} \tag{31}$$

The Young operator $\mathcal{Y}_1^{[3,2]}$ is (see (6.23) in [28])

$$\mathcal{Y}_1^{[3,2]} = [E + (1\,2) + (1\,3) + (2\,3) + (1\,2\,3) + (3\,2\,1)][E + (4\,5)][E - (1\,4)][E - (2\,5)]. \tag{32}$$

The spinor function ϕ_1 with $S_z = S = 1/2$ in the tensor subspace $\mathcal{T}_1^{[3,2]}$ is denoted by the tensor Young tableau (12)

$$\begin{aligned} \phi_1 = \mathcal{Y}_1^{[3,2]} Z_1 = 12\boldsymbol{\theta}_{11122} - 4\{&\boldsymbol{\theta}_{11212} + \boldsymbol{\theta}_{12112} + \boldsymbol{\theta}_{21112} + \boldsymbol{\theta}_{11221} + \boldsymbol{\theta}_{12121} + \boldsymbol{\theta}_{21121}\} \\ &+ 4\{\boldsymbol{\theta}_{12211} + \boldsymbol{\theta}_{22111} + \boldsymbol{\theta}_{21211}\}, \end{aligned} \tag{33}$$

where $Z_1 = \boldsymbol{\theta}_{11122}$. $\phi_\mu = R_{\mu 1}\phi_1$ is the basis tensor in the tensor subspaces $\mathcal{T}_\mu^{[3,2]}$, denoted by the same tensor Young tableau (12). The spinor functions ϕ_μ span the representation space of [3, 2] of S_5. From (31), we have $R_{11} = E$, $R_{21} = (3\,4)$, $R_{31} = (3\,5\,4)$, $R_{41} = (2\,3\,4)$ and $R_{51} = (2\,3\,5\,4)$.

The spatial wave function $\psi_1 = \mathcal{Y}_1^{[3,2]} Q_1$, where $Q_1 = Q_{4,5}^{1,2,3}$, is

$$\begin{aligned} \psi_1 = \mathcal{Y}_1^{[3,2]} Q_1 &= 12 Q_{4,5}^{1,2,3} - 4\big(Q_{3,5}^{1,2,4} + Q_{2,5}^{1,3,4} + Q_{1,5}^{2,3,4} + Q_{3,4}^{1,2,5} + Q_{2,4}^{1,3,5} + Q_{1,4}^{2,3,5}\big) \\ &\quad + 4\big(Q_{2,3}^{1,4,5} + Q_{1,2}^{3,4,5} + Q_{1,3}^{2,4,5}\big) && (34) \\ &= 4[3E - (3\,4) - (2\,4) - (1\,4) - (3\,5) - (2\,5) - (1\,5) && (35) \\ &\quad + (2\,4)(3\,5) + (1\,4)(2\,5) + (1\,4)(3\,5)] Q_{4,5}^{1,2,3}. && (36) \end{aligned}$$

$\psi_\mu = R_{\mu 1}\psi_1$ span the representation space of [3, 2] of S_5.

The similarity transformation $X_{[3,2]}$ from the representation in the standard bases to that in the orthogonal bases is given in (6.92) of [28]:

$$X_{[3,2]} = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{8} & 1 & \sqrt{3} & \sqrt{3} & 3 \\ 0 & 3 & \sqrt{3} & \sqrt{3} & 1 \\ 0 & 0 & 2\sqrt{3} & 0 & 2 \\ 0 & 0 & 0 & 2\sqrt{3} & 2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}. \tag{37}$$

Then, neglecting the normalization factor, we have

$$\begin{aligned} \psi_S^{[3,2]} = \psi_1&[6\phi_1 + 3\phi_2 + 3\phi_3 + 3\phi_4 + 3\phi_5] + \psi_2[3\phi_1 + 4\phi_2 + 2\phi_3 + 2\phi_4 + \phi_5] \\ &+ \psi_3[3\phi_1 + 2\phi_2 + 4\phi_3 + \phi_4 + 2\phi_5] + \psi_4[3\phi_1 + 2\phi_2 + \phi_3 + 4\phi_4 + 2\phi_5] \\ &+ \psi_5[3\phi_1 + \phi_2 + 2\phi_3 + 2\phi_4 + 4\phi_5]. \end{aligned} \tag{38}$$

Substituting (36) into (38), we obtain the expression of $\psi_S^{[3,2]}$ by collecting terms with the same $Q_{b_1,b_2}^{a_1,a_2,a_3}$,

$$\begin{aligned} \psi_S^{[3,2]} = 24\big\{&Q_{4,5}^{1,2,3}\phi_1 + Q_{3,5}^{1,2,4}\phi_2 + Q_{3,4}^{1,2,5}\phi_3 + Q_{2,5}^{1,3,4}\phi_4 + Q_{2,4}^{1,3,5}\phi_5 \\ &- Q_{2,3}^{1,4,5}[\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5] - Q_{1,5}^{2,3,4}[\phi_1 + \phi_2 + \phi_4] \\ &- Q_{1,4}^{2,3,5}[\phi_1 + \phi_3 + \phi_5] + Q_{1,3}^{2,4,5}[\phi_1 + \phi_4 + \phi_5] + Q_{1,2}^{3,4,5}[\phi_1 + \phi_2 + \phi_3]\big\}. \end{aligned} \tag{39}$$

Now, in terms of the fundamental property of the Young operators (see section 6.2.4 in [28])

$$(1\,2)\phi_1 = (1\,3)\phi_1 = (2\,3)\phi_1 = (4\,5)\phi_1 = \phi_1,$$
$$[E + (1\,4) + (2\,4) + (3\,4)]\,\phi_1 = [E + (1\,5) + (2\,5) + (3\,5)]\,\phi_1 = 0,$$

we have

$$\phi_2 = (3\,4)\phi_1, \qquad \phi_3 = (3\,5)\phi_1, \qquad \phi_4 = (2\,4)\phi_1, \qquad \phi_5 = (2\,5)\phi_1,$$
$$(2\,4)(3\,5)\phi_1 = (3\,4)(2\,5)\phi_1, \qquad (1\,4)(3\,5)\phi_1 = (3\,4)(1\,5)\phi_1,$$
$$(1\,4)(2\,5)\phi_1 = (2\,4)(1\,5)\phi_1$$

and

$$\begin{aligned}(2\,4)(3\,5)\phi_1 &= -(2\,4)\,[E + (1\,5) + (2\,5)]\,\phi_1 \\ &= -(2\,4)\phi_1 - (2\,5)\phi_1 + (1\,5)\,[E + (1\,4) + (3\,4)]\,\phi_1 \\ &= -\,[(2\,4) + (2\,5) - (1\,4) - (1\,5)]\,\phi_1 - (3\,4)\,[E + (2\,5) + (3\,5)]\,\phi_1 \\ &= -\,[(2\,4) + (2\,5) + (3\,4) + (3\,5) - (1\,4) - (1\,5)]\,\phi_1 - (3\,4)(2\,5)\phi_1 \\ &= -\,[E + (2\,4) + (2\,5) + (3\,4) + (3\,5)]\,\phi_1.\end{aligned}$$

Then,

$$(2\,4)(1\,5)\phi_1 = [E + (3\,4) + (3\,5)]\phi_1,$$
$$(3\,4)(1\,5)\phi_1 = [E + (2\,4) + (2\,5)]\phi_1.$$

Substituting them into (39), we obtain

$$\begin{aligned}\psi_S^{[3,2]} &= 24[E + (3\,4) + (3\,5) + (2\,4) + (2\,5) + (2\,4)(3\,5) \\ &\quad + (1\,4) + (1\,5) + (1\,4)(3\,5) + (1\,4)(2\,5)]Q_1\big(\mathcal{Y}_1^{[3,2]}Z_1\big).\end{aligned} \tag{40}$$

6. Representation [2, 2]

We discuss the symmetric function $\psi_S^{[2,2]}$ with $N = 4$ and $S = S_z = 0$ as the third example. There are two standard Young tableaux for the Young pattern [2, 2]:

$$\begin{array}{cc} \mathcal{Y}_1^{[2,2]} & \mathcal{Y}_2^{[2,2]} \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\end{array} \tag{41}$$

The Young operator $\mathcal{Y}_1^{[2,2]}$ is (see (6.23) in [28])

$$\mathcal{Y}_1^{[2,2]} = [E + (1\,2)][E + (3\,4)][E - (1\,3)][E - (2\,4)]. \tag{42}$$

The spinor function ϕ_μ with $S_z = S = 0$ in the tensor subspace $\mathcal{T}_\mu^{[2,2]}$ ($\mu = 1$ and 2), respectively, is

$$\begin{aligned}\phi_1 &= \mathcal{Y}_1^{[2,2]}Z_1 = 4\boldsymbol{\theta}_{1122} - 2\,\{\boldsymbol{\theta}_{2112} + \boldsymbol{\theta}_{1212} + \boldsymbol{\theta}_{2121} + \boldsymbol{\theta}_{1221}\} + 4\boldsymbol{\theta}_{2211}, \\ \phi_2 &= (2\,3)\phi_1 = 4\boldsymbol{\theta}_{1212} - 2\,\{\boldsymbol{\theta}_{2112} + \boldsymbol{\theta}_{1122} + \boldsymbol{\theta}_{2211} + \boldsymbol{\theta}_{1221}\} + 4\boldsymbol{\theta}_{2121},\end{aligned} \tag{43}$$

where $Z_1 = \boldsymbol{\theta}_{1122}$. Both the spinor functions ϕ_1 and ϕ_2 are denoted by the tensor Young tableau (12) and they span the two-dimensional representation space of [2, 2] of S_4.

The spatial wave functions $\psi_1 = \mathcal{Y}_1^{[2,2]}Q_1$ and $\psi_2 = (2\,3)\psi_1$ are

$$\begin{aligned}\psi_1 &= \mathcal{Y}_1^{[2,2]}Q_1 = 8Q_{3,4}^{1,2} - 4Q_{2,4}^{1,3} - 4Q_{2,3}^{1,4}, \\ \psi_2 &= (2\,3)\psi_1 = 8Q_{2,4}^{1,3} - 4Q_{3,4}^{1,2} - 4Q_{2,3}^{1,4},\end{aligned} \tag{44}$$

where $Q_1 = Q^{1,2}_{3,4} = Q^{3,4}_{1,2}$. They span the representation space of [2, 2] of S_4.

The similarity transformation $X_{[2,2]}$ from the representation in the standard bases to that in the orthogonal bases is given in problem 24 of chapter 6 in [29]:

$$X_{[2,2]} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix}. \tag{45}$$

Then, we have

$$\psi_S^{[2,2]} = \tfrac{2}{3}[\psi_1(2\phi_1 + \phi_2) + \psi_2(\phi_1 + 2\phi_2)]. \tag{46}$$

Substituting (43) and (44) into (46), we obtain the expression of $\psi_S^{[2,2]}$ by collecting terms with the same $Q^{a_1,a_2}_{b_1,b_2}$,

$$\begin{aligned}\psi_S^{[2,2]} &= 2\big[Q^{1,2}_{3,4}\phi_1 + Q^{1,3}_{2,4}\phi_2 - Q^{1,4}_{2,3}(\phi_1 + \phi_2)\big] \\ &= [E + (1\,3)(2\,4) + (1\,4) + (2\,3) + (1\,3) + (2\,4)]Q^{1,2}_{3,4}\big(\mathcal{Y}_1^{[2,2]}Z_1\big),\end{aligned} \tag{47}$$

where the Fock condition (see (6.30) in [28]) is used:

$$\begin{aligned}-(1\,4)\mathcal{Y}_1^{[2,2]} &= -(2\,3)\mathcal{Y}_1^{[2,2]} = [E + (2\,4)]\,\mathcal{Y}_1^{[2,2]}, \\ -(1\,3)\mathcal{Y}_1^{[2,2]} &= -(2\,4)\mathcal{Y}_1^{[2,2]} = [E + (2\,3)]\,\mathcal{Y}_1^{[2,2]}.\end{aligned}$$

7. Symmetric function

We have calculated the symmetric function $\psi_S^{[4,1]}$ (30) for $N = 5$, $S_z = S = 3/2$, $\psi_S^{[3,2]}$ (40) for $N = 5$ and $S_z = S = 1/2$, and $\psi_S^{[2,2]}$ (47) for $N = 4$ and $S_z = S = 0$. We now analyze the common property of the symmetric functions $\psi_S^{[n,m]}$, find the general form of $\psi_S^{[n,m]}$ and prove it.

Let $B_\alpha = \{b_1, b_2, \ldots, b_m\}$ be a set of m different integers where $1 \leqslant b_1 < b_2 < \cdots < b_m \leqslant N$. The $n = N - m$ remaining different integers $a_1, a_2, \cdots, a_n$, satisfying $a_i \neq b_j$ and $1 \leqslant a_1 < a_2 < \ldots < a_n \leqslant N$ are also determined by the set B_α. There are $N!/(n!m!)$ different sets B_α. Assume that $a_i = i$ and $b_j = n + j$ when $\alpha = 1$. Corresponding to a set B_α, we define a permutation P_α,

$$P_\alpha = \begin{pmatrix} 1 & 2 & \ldots & n & n+1 & n+2 & \ldots & N \\ a_1 & a_2 & \ldots & a_n & b_1 & b_2 & \ldots & b_m \end{pmatrix}. \tag{48}$$

Recall that P_1 is the identical permutation. The left coset of $H_{nm} = \mathrm{S}_n \otimes \mathrm{S}_m$ in S_N is denoted by $P_\alpha H_{nm}$.

For a given Young pattern $[n, m]$, define $N!/(n!m!)$ different Young operators $\mathcal{Y}_\alpha^{[n,m]}$ which are generally not standard,

$$\mathcal{Y}_\alpha^{[n,m]} = P_\alpha \mathcal{Y}_1^{[n,m]} P_\alpha^{-1}. \tag{49}$$

In fact, $\mathcal{Y}_\alpha^{[n,m]}$ corresponds to the Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline a_1 & \ldots & a_m & a_{m+1} & \ldots & a_n \\ \hline b_1 & \ldots & b_m \\ \cline{1-3}\end{array}\,.$$

Thus,

$$Z_\alpha = P_\alpha Z_1, \qquad P_\alpha \mathcal{Y}_1^{[n,m]} Z_1 = \mathcal{Y}_\alpha^{[n,m]} Z_\alpha, \tag{50}$$

$$Q_\alpha = P_\alpha Q_1 = Q^{a_1,\ldots,a_n}_{b_1,\ldots,b_m} = \prod_{j=1}^{n}\prod_{k=1}^{m} \mathrm{sgn}(x_{a_j} - x_{b_k}). \tag{51}$$

Then, (30), (40) and (47) can be rewritten in a unified form as given in the following theorem.

Theorem. *The totally symmetric wave function constructed by the product of the sign functions* Q_α *and the basis tensors* Z_α *is uniquely expressed as*

$$\psi_S^{[n,m]} = C\left\{\sum_{\alpha=1}^{N!/(n!m!)} P_\alpha\right\}\left\{Q_1\left(\mathcal{Y}_1^{[n,m]} Z_1\right)\right\} = C\sum_{\alpha=1}^{N!/(n!m!)} Q_\alpha\left(\mathcal{Y}_\alpha^{[n,m]} Z_\alpha\right), \tag{52}$$

where C *is the normalization factor.*

Proof. Since Q_1 and $\mathcal{Y}_1^{[n,m]} Z_1$ both are invariant in the permutations of the subgroup $\mathrm{S}_n \otimes \mathrm{S}_m$, the action of $\sum_\alpha P_\alpha$ is proportional to that of the sum over all elements in S_N

$$\left\{\sum_{\alpha=1}^{N!/(n!m!)} P_\alpha\right\}\left\{Q_1\left(\mathcal{Y}_1^{[n,m]} Z_1\right)\right\} = \frac{1}{n!m!}\left\{\sum_{R\in \mathrm{S}_N} R\right\}\left\{Q_1\left(\mathcal{Y}_1^{[n,m]} Z_1\right)\right\},$$

so that $\psi_S^{[n,m]}$ is invariant in S_N. Q_1 in (21) spans the representation space of the identical representation $[n]\times[m]$ of $\mathrm{S}_n\otimes \mathrm{S}_m$ so that $P_\alpha Q_1 = Q_\alpha$ spans its induced representation $[n]\otimes[m]$ with respect to S_N which can be decomposed by the Littlewood–Richardson rule (see section 6.5 in [28])

$$[n]\otimes[m] \simeq [N]\oplus[N-1,1]\oplus\cdots\oplus[n,m]. \tag{53}$$

Since the multiplicity of $[n,m]$ in the reduction (53) is 1, the symmetrized function (52) is unique if the spatial functions are constructed by the sign functions in the form (51). □

As a by-product, the spatial function corresponding to the identical representation $[N]$ in (53) provides an identical equation

$$\sum_{\alpha=1}^{N!/(n!m!)} Q_\alpha = \text{const}. \tag{54}$$

The constant can be counted in any section, say $-\infty < x_N < x_{N-1} < \cdots < x_1 < \infty$. When $n=m$ is even, $Q^{a_1,\ldots,a_m}_{b_1,\ldots,b_m} = Q^{b_1,\ldots,b_m}_{a_1,\ldots,a_m}$, and the identity (54) is simplified by a factor 2. When $n=m$ is odd, $Q^{a_1,\ldots,a_m}_{b_1,\ldots,b_m} = -Q^{b_1,\ldots,b_m}_{a_1,\ldots,a_m}$, and the identity (54) becomes trivial.

Collecting terms with the same Z_α in (52), we obtain another form for $\psi_S^{[n,m]}$:

$$\psi_S^{[n,m]} = C\sum_{\alpha=1}^{N!/(n!m!)} \left(\mathcal{Y}_\alpha^{[n,m]} Q_\alpha\right) Z_\alpha. \tag{55}$$

For example, neglecting the normalization factor, we have

$$\begin{aligned}
\psi_S^{[2,1]} &= \left(2Q_3^{1,2} - Q_1^{2,3} - Q_2^{1,3}\right)\boldsymbol{\theta}_{112} + \left(2Q_2^{1,3} - Q_1^{2,3} - Q_3^{1,2}\right)\boldsymbol{\theta}_{121} \\
&\quad + \left(2Q_1^{2,3} - Q_3^{1,2} - Q_2^{1,3}\right)\boldsymbol{\theta}_{211}, \\
&= \left(3Q_3^{1,2} - 1\right)\boldsymbol{\theta}_{112} + \left(3Q_2^{1,3} - 1\right)\boldsymbol{\theta}_{121} + \left(3Q_1^{2,3} - 1\right)\boldsymbol{\theta}_{211}, \\
&\quad Q_3^{1,2} + Q_2^{1,3} + Q_1^{2,3} = 1, \\
\psi_S^{[3,1]} &= Q_4^{1,2,3}\boldsymbol{\theta}_{1112} + Q_3^{1,2,4}\boldsymbol{\theta}_{1121} + Q_2^{1,3,4}\boldsymbol{\theta}_{1211} + Q_1^{2,3,4}\boldsymbol{\theta}_{2111}, \\
&\quad Q_4^{1,2,3} + Q_3^{1,2,4} + Q_2^{1,3,4} + Q_1^{2,3,4} = 0,
\end{aligned}$$

$$\begin{aligned}
\psi_S^{[2,2]} &= \left(3Q_{3,4}^{1,2}-1\right)(\boldsymbol{\theta}_{1122}+\boldsymbol{\theta}_{2211})+\left(3Q_{2,4}^{1,3}-1\right)(\boldsymbol{\theta}_{1212}+\boldsymbol{\theta}_{2121})\\
&\quad+\left(3Q_{2,3}^{1,4}-1\right)(\boldsymbol{\theta}_{1221}+\boldsymbol{\theta}_{2112}),\\
&\quad Q_{3,4}^{1,2}+Q_{2,4}^{1,3}+Q_{2,3}^{1,4}=1,\\
\psi_S^{[4,1]} &= \left(5Q_5^{1,2,3,4}-1\right)\boldsymbol{\theta}_{11112}+\left(5Q_4^{1,2,3,5}-1\right)\boldsymbol{\theta}_{11121}+\left(5Q_3^{1,2,4,5}-1\right)\boldsymbol{\theta}_{11211}\\
&\quad+\left(5Q_2^{1,3,4,5}-1\right)\boldsymbol{\theta}_{12111}+\left(5Q_1^{2,3,4,5}-1\right)\boldsymbol{\theta}_{21111},\\
&\quad Q_5^{1,2,3,4}+Q_4^{1,2,3,5}+Q_3^{1,2,4,5}+Q_2^{1,3,4,5}+Q_1^{2,3,4,5}=1,\\
\psi_S^{[3,2]} &= \sum_\alpha P_\alpha\left\{\left(2Q_{4,5}^{1,2,3}+Q_{1,2}^{3,4,5}+Q_{1,3}^{2,4,5}+Q_{2,3}^{1,4,5}-1\right)\theta_{11122}\right\},\\
P_\alpha &= \begin{pmatrix}1&2&3&4&5\\a_1&a_2&a_3&b_1&b_2\end{pmatrix},\qquad \sum_{b_1=1}^{4}\sum_{b_2=b_1+1}^{5}Q_{b_1,b_2}^{a_1,a_2,a_3}=2.
\end{aligned}$$

The symmetric function with $S = N/2 - m$ and $S_z = S - \tau$ is

$$\begin{aligned}
\psi_S^{[n,m],S_z} &= C_\tau\left\{\sum_{\alpha=1}^{N!/(n!m!)}P_\alpha\right\}\left\{Q_1\left(\mathcal{Y}_1^{[n,m]}Z_1^{(\tau)}\right)\right\}=C_\tau\sum_{\alpha=1}^{N!/(n!m!)}Q_\alpha\left(\mathcal{Y}_\alpha^{[n,m]}Z_\alpha^{(\tau)}\right)\\
&\neq C_\tau\sum_{\alpha=1}^{N!/(n!m!)}\left(\mathcal{Y}_\alpha^{[n,m]}Q_\alpha\right)Z_\alpha^{(\tau)},
\end{aligned}\tag{56}$$

where $Z_1^{(\tau)}$ is given in (13), $Z_\alpha^{(\tau)} = P_\alpha Z_1^{(\tau)}$ and the normalization factor C_τ may be different from C.

8. Density distributions

The spin-dependent reduced one-body density matrices $\rho_\sigma^{(S,S_z)}(x,x')$ are defined as

$$\rho_\sigma^{(S,S_z)}(x,x') = \int\left[\psi_A(x,X)\psi_S^{[n,m],S_z}(x,X)\right]^\dagger P_\sigma^{(1)}\left[\psi_A(x',X)\psi_S^{[n,m],S_z}(x',X)\right]\prod_{i=2}^N \mathrm{d}x_i,\tag{57}$$

where $X = (x_2,\cdots,x_N)$, $P_\sigma^{(1)} = 1/2 - (-1)^\sigma S_z^{(1)}$ with $\sigma = 1$ or 2. $\rho_\sigma^{(S,S_z)}(x,x')$ evidently depends on the spin $S = N/2 - m$ and $S_z = S - \tau$. The spin-dependent single particle densities $\rho_\sigma^{(S,S_z)}(x) = \rho_\sigma^{(S,S_z)}(x,x)$ are the diagonal elements of the corresponding reduced density matrices,

$$\rho_\sigma^{(S,S_z)}(x) = \int\left[\psi_A(x,X)\psi_S^{[n,m],S_z}(x,X)\right]^\dagger P_\sigma^{(1)}\left[\psi_A(x,X)\psi_S^{[n,m],S_z}(x,X)\right]\prod_{i=2}^N \mathrm{d}x_i,\tag{58}$$

and satisfy the normalized conditions

$$\int \mathrm{d}x\,\rho_\sigma^{(S,S_z)}(x) = N_\sigma,\tag{59}$$

where $N_1 = N - m - \tau$ and $N_2 = N - N_1$ are the numbers of two-component upspinors and downspinors, respectively. Interchanging $1 \leftrightarrow 2$ for all subscripts σ_i of $\chi_{\sigma_i}(i)$ in $\psi_S^{[n,m],S_z}(x,X)$ and for σ in $P_\sigma^{(1)}$, from (58) and (17) one obtains

$$\rho_1^{(S,S_z)}(x) = \rho_2^{(S,-S_z)}(x).\tag{60}$$

The total density is defined as $\rho^{(S)}(x) = \rho_1^{(S,S_z)}(x) + \rho_2^{(S,S_z)}(x)$, which may depend on the total spin S but must be independent of S_z owing to the symmetry of $SU(2)$. For N-fermion ferromagnetic system, $S = N/2$, the spinor wave function is totally symmetric with respect to any permutation among fermions such that

$$\rho_\sigma^{(S,S_z)}(x) = \frac{N_\sigma}{N}\rho^{(S)}(x), \qquad \text{when} \quad S = N/2. \tag{61}$$

For $S < N/2$, $\rho_\sigma^{(S,S_z)}(x)$ are generally not proportional to each other. However, when $N_1 = N_2 = N/2$ ($S_z = 0$), from (60) one has

$$\rho_1^{(S,0)}(x) = \rho_2^{(S,0)}(x) = \tfrac{1}{2}\rho^{(S)}(x). \tag{62}$$

We now prove that the total density $\rho^{(S)}(x)$ of the exact solution $\psi = \psi_A\psi_S^{[n,m]}$ is independent of the total spin S such that it is equal to the density $\rho_{fm}(x) = \rho^{(N/2)}(x)$ of an N-fermion ferromagnetic system, which is also called the density of a polarized free N-fermion system,

$$\rho_{fm}(x) = N\int \psi_A(x,X)^*\psi_A(x,X)\prod_{i=2}^{N}\mathrm{d}x_i = \sum_{i=0}^{N-1}|\phi_i(x)|^2, \tag{63}$$

where $\phi_i(x)$ is explained in equation (3).

Proof. We divide the whole space Ω into $N!$ sections Ω_R,

$$\begin{aligned} &\Omega = \bigcup_{R\in S_N}\Omega_R, \qquad \Omega_R = R\Omega_E, \\ &\Omega_E = \{-\infty < x_N < x_{N-1} < \cdots < x_1 < \infty\}. \end{aligned} \tag{64}$$

In each section Ω_R, Q_α as well as $\mathcal{Y}_\alpha Q_\alpha$ is a constant:

$$\mathcal{Y}_\alpha Q_\alpha|_{\Omega_R} = A_{R\alpha}. \tag{65}$$

As proved in the theorem, $\mathcal{Y}_1 Q_1$ is invariant in the subgroup $S_n \otimes S_m$, and $\sum_\alpha(\mathcal{Y}_\alpha Q_\alpha)^2$ is invariant in the permutation group S_N. Namely,

$$\sum_{\alpha=1}^{N!/(n!m!)}(\mathcal{Y}_\alpha Q_\alpha)^2 = \text{const} \tag{66}$$

since it is a constant independent of x such that

$$\sum_{\alpha=1}^{N!/(n!m!)}(\mathcal{Y}_\alpha Q_\alpha)^2 = \sum_{\alpha=1}^{N!/(n!m!)}(A_{R\alpha})^2 = \text{const}. \tag{67}$$

Thus, from (58) one has

$$\begin{aligned} \rho^{(S)}(x) &= \int\left[\psi_A(x,X)\psi_S^{[n,m]}(x,X)\right]^\dagger\left[\psi_A(x,X)\psi_S^{[n,m]}(x',X)\right]\prod_{i=2}^{N}\mathrm{d}x_i \\ &= |C|^2\int\left[\sum_{\alpha=1}^{N!/(n!m!)}(\mathcal{Y}_\alpha Q_\alpha)^2\right]\psi_A(x,X)^*\psi_A(x,X)\prod_{i=2}^{N}\mathrm{d}x_i \\ &= |C|^2\left[\sum_{\alpha=1}^{N!/(n!m!)}(\mathcal{Y}_\alpha Q_\alpha)^2\right]N^{-1}\rho_{fm}(x). \end{aligned} \tag{68}$$

The normalized condition for $\rho^{(S)}(x)$ is

$$\begin{aligned} N &= \int \rho(x)\,\mathrm{d}x \\ &= |C|^2 \sum_{R\in S_N} \left[\sum_{\alpha=1}^{N!/(n!m!)} (A_{R\alpha})^2 \right] \int_{\Omega_R} N^{-1}\rho_{fm}(x)\,\mathrm{d}x \\ &= |C|^2 \sum_{\alpha=1}^{N!/(n!m!)} (A_{R\alpha})^2. \end{aligned}$$

Thus, from (67), we come to the conclusion

$$\rho(x) = \rho_{fm}(x). \tag{69}$$

□

Finally, we would like to discuss the meaning of (62). For an equal-mixing Fermi gas in an arbitrary confining potential, say a harmonic trap, the density distribution of each component displays N peaks in the infinitely interacting limit. However, for cases with $N_1 \neq N_2$, no simple analytical expression can be obtained despite the total density distribution having a simple expression. As a comparison, we note that for a two-component Bose system with $SU(2)$ symmetry, the density distribution of each component is proportional to the total density distribution, i.e. $\rho_i(x) = \frac{N_i}{N}\rho(x)$ [20, 30]. The intrinsic difference of the spin-dependent density distribution between Fermi and Boson systems is attributed to their different exchange symmetry of the ground state wavefunction. As a limiting case of the strong coupling Fermi system, our exact result provides a firm ground for various methods, for example, the density functional theory in combination with the local density approximation which has been widely applied to the study of the spin-1/2 Fermi gas [23–26].

9. Other solutions for ψ_S

In section 3, we have assumed that the spatial part of the symmetric wave function ψ_S is composed by the sign functions in the form (21). However, the form (21) is not the only one that can construct the spatial part of ψ_S by the sign functions. We will discuss the general form of the symmetric wave function ψ_S constructed by the product of sign functions and spinor functions.

The whole space Ω is divided into $N!$ sections Ω_R as given in (64). Define the $N!$ group functions $f_R(x)$:

$$f_R(x)|_{\Omega_S} = \begin{cases} 1 & S = R, \\ 0 & S \neq R, \end{cases} \qquad T f_R(x) = f_{TR}(x). \tag{70}$$

f_R spans the representation space of the regular representation of S_N. Any product of sign functions is a linear combination of f_R because a sign function is a constant in any section Ω_R. Conversely, f_R can be expressed as a linear combination of the products of sign functions, including 1 which is the 'product' of sign functions of zero power. Let us demonstrate this conclusion by a simple example. Ω is divided into six sections Ω_R when $N = 3$:

$$\begin{aligned} &\Omega_E: && -\infty < x_3 < x_2 < x_1 < \infty, \\ &\Omega_A: && -\infty < x_2 < x_3 < x_1 < \infty, \\ &\Omega_B: && -\infty < x_1 < x_2 < x_3 < \infty, \\ &\Omega_C: && -\infty < x_3 < x_1 < x_2 < \infty, \\ &\Omega_D: && -\infty < x_1 < x_3 < x_2 < \infty, \\ &\Omega_F: && -\infty < x_2 < x_1 < x_3 < \infty. \end{aligned}$$

Then, we have

$$\begin{aligned}
f_E &= \tfrac{1}{4}(\mathrm{sgn}_{12}+\mathrm{sgn}_{23})(\mathrm{sgn}_{13}+1), & f_B &= \tfrac{1}{4}(\mathrm{sgn}_{12}+\mathrm{sgn}_{23})(\mathrm{sgn}_{13}-1),\\
f_A &= \tfrac{1}{4}(\mathrm{sgn}_{13}-\mathrm{sgn}_{23})(\mathrm{sgn}_{12}+1), & f_D &= \tfrac{1}{4}(\mathrm{sgn}_{13}-\mathrm{sgn}_{23})(\mathrm{sgn}_{12}-1),\\
f_C &= \tfrac{1}{4}(\mathrm{sgn}_{13}-\mathrm{sgn}_{12})(\mathrm{sgn}_{23}+1), & f_F &= \tfrac{1}{4}(\mathrm{sgn}_{13}-\mathrm{sgn}_{12})(\mathrm{sgn}_{23}-1).
\end{aligned}$$

Define $M(n,m) = N!/(n!m!)$ functions $F_\beta(x,n,m)$ which are invariant with respect to $H_{nm} = \mathrm{S}_n \otimes \mathrm{S}_m$,

$$\begin{aligned}
&F_\beta(x,n,m) = \sum_{R\in H_{nm}P_\beta} f_R(x), \qquad \beta = 1,2,\ldots,M(n,m),\\
&T F_\beta(x,n,m) = F_\beta(x,n,m), \qquad \text{when} \quad T \in H_{nm},\\
&F_\beta(x,n,m)\big|_{\Omega_R} = \begin{cases} 1 & R \in H_{nm}P_\beta,\\ 0 & \text{otherwise},\end{cases}
\end{aligned} \tag{71}$$

where P_β is given in (48). There are $M(n,m)$ symmetric functions ψ_S with $S \geqslant N/2 - m = (n-m)/2$, which are linearly independent to each other,

$$\begin{aligned}
&\sum_{\alpha=1}^{M(n,m)} P_\alpha[F_\beta(x,n,m)Z_1] = \sum_{\alpha=1}^{M(n,m)} [P_\alpha F_\beta(x,n,m)]Z_\alpha,\\
&P_\alpha F_\beta(x,n,m) = \sum_{R\in P_\alpha H_{nm}P_\beta} f_R(x),
\end{aligned} \tag{72}$$

where Z_1 and Z_α are given in (11) and (50), respectively. Among them there are $M(n+1,m-1)$ functions with $S > N/2 - m$ such that the remaining $M(n,m) - M(n+1,m-1)$ functions with $S = N/2 - m$. Note that

$$M(n,m) - M(n+1,m-1) = \frac{N!\,(n+1-m)}{(n+1)!m!} = d_{[n,m]}(\mathrm{S}_N).$$

In other words, there are only $d_{[n,m]}(\mathrm{S}_N)$ symmetric functions ψ_S with $S = S_z = N/2 - m$ in the following forms which are linearly independent of each other:

$$\begin{aligned}
&\sum_{\alpha=1}^{M(n,m)} P_\alpha\left[\Phi_t(x,n,m)\mathcal{Y}_1^{[n,m]}Z_1\right], \qquad 1 \leqslant t \leqslant d_{[n,m]}(\mathrm{S}_N),\\
&\Phi_t(x,n,m) = \sum_{\beta=1}^{M(n,m)} B_{\beta t}F_\beta(x,n,m),
\end{aligned} \tag{73}$$

where $\Phi_1(x,n,m) = Q_1$ given in the symmetric function (52). Q_1 is the only $\Phi_t(x,n,m)$ in (73) whose square is equal to 1 in any section Ω_R:

$$Q_1^2\big|_{\Omega_R} = 1, \qquad \forall R. \tag{74}$$

This may be the reason why our solution (52) is a good approximation of the true wave function when the interaction strength is very large but not infinite as shown in the numerical calculations [1].

10. Summary

We have constructed exact solutions of the fundamental system of quasi-1D spin-1/2 fermions with infinite δ repulsion by means of the group theoretical method in some detail. The exact solutions are the simultaneous eigenstates of the Hamiltonian H and the total spin operators S^2

and S_z, which fulfil Girardeau's hard-core contacting boundary condition, are antisymmetric under odd permutations among fermions and are unique if the additional spatial functions are constructed by the sign functions in the form (51). Since we have given the general scheme for the construction of a state with arbitrary S, it is easy to choose the state with lowest S which is expected to be a good description of the ground state even for a system with large but not infinite repulsion [1, 31][4]. We also prove that the total GS density $\rho(x)$ is equal to the density of an N-fermion ferromagnetic system. Since the exact construction of the ground state wave function has been given, it would be interesting to apply the exact wave function to study the ground state properties and correlation functions in this system directly. This, however, remains a difficult task for a large system due to the time consumed in calculating multidimensional integrals and is beyond the scope of the present work.

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Note added in proof. After the paper was completed we read a preprint by C N Yang (2009, arXiv:0906.4593) in which it is proved (see (9) therein) that the solution (52) is only an approximation of the true wave function when the interaction strength is very large but not infinite.

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